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The composite anyon fields are shown to satisfy the proper anyonic commutation relations with the additive phase exponents. Then, quasiparticle picture of the anyons is clarified under the restriction of this additivity. The difference between field and particle aspects becomes more prominent in the 2 space dimension. It is argued that the hierarchy of the fractional quantum Hall effect is rather simply understood by utilizing the quasiparticle characters of the anyons when the background-boson gauge is assumed. In contrast to it, the composite fermion theories are critically reviewed.

Statistics of the Composite Systems
and
Anyons in the Fractional Quantum Hall Effect

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§1. Introduction

More than forty years ago, composite systems in the quantum field theory was investigated. One starts from a local scalar field, say $A(x)$, and assumes the existence of discrete eigenvalues m^2 and M^2 of the 4-momentum squared P^2 , where m is the mass of the original field A . If $\langle A(x)A(y) \mid P^2 = M^2 \rangle \neq 0$ there may be a composite(bound) state of the mass M . Then, one defines a bilocal field

$$B(x, \varepsilon) = T A(x + \varepsilon) A(x - \varepsilon) \quad (1)$$

representing this state, where T denotes the time-ordered product. Zimmermann showed, under some mathematical assumption, that the asymptotic($t \rightarrow \pm\infty$) field of B satisfies the proper commutation relation in the limit $\varepsilon \rightarrow 0$, if it is suitably normalized¹⁾. One can infer, from this construction, that composite systems can be described by the local field operators and there are no differences between elementary and composite particles in constructing the S matrix elements. An idea of the boot strapping(nuclear democracy) emerged from this observation.

However, the successes of the gauge theories have changed drastically the framework of the elementary particle theory. The quantum theory of field has revived and one gradually recognized the hierarchy structure of the gauge interactions. A field theory now becomes an effective theory for each class of the hierarchy.

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On the other hand, there is another hierarchy of compositeness in nature²⁾, which was revealed through success of the composite models of elementary particles. The hierarchy here consists of the classes of quark, hadron, nucleus, atom and so on. The level of a class is specified by the energy scale indicating the limit of applicability of the theory governing it. We, further, believe that the theory is a quantum field theory and there exist elementary fields for each class, which are constructed from the elementary fields of the deeper class. The most instructive example of this interpolating mechanism may be provided by considering the class of atom. An atom is a composite system which consists of a nucleus field and an electron field interacting through the photon field. Then, we construct the atom field as a composite field of the elementary fields of the deeper class. In this respect, we are specially interested in the statistic property of the composite system, since we are hardly convinced of it by the particle quantum mechanics: For example, the nucleus of the hydrogen atom is a Fermi particle and it changes into the Bose particle by acquiring another fermion. This is mysterious from the view point of the particle quantum mechanics^{**)}.

The difference between the quantum field theory and the many-particle quantum mechanics is more prominent in the world of the 2 space dimension. Its topological structure allows the exotic statistics of the field operators, which cannot always be realized by the flux-charge composite having the characteristics of (quasi)particle. Only if we can define attributes of it consistently we call it the quasiparticle anyon. The first purpose of the present paper is to give a unified consideration to the essential points of the field-particle duality in the composite systems. We also emphasize the topological difference between 3 and 2 space-dimensions. The main issue is how to understand the Chern-Simons(CS) gauge field which governs the statistics of the 2-dimensional space. We consider it to be the field of a boundary condition to be eliminated in the final physical results.

The complicated nature of the statistics of the anyons sometimes confuses theoretical understanding of the related phenomena. An example is the fractional quantum Hall effect(FQHE). Our second purpose is to clarify the hierarchy structure of the fractional quantum Hall state(FQHS) from the point of view of the CS gauge theory of the anyons, which has not always been recognized correctly in this research field.

We study the field aspects of the hydrogen atom in §2 and show the canonical commutation relations for the atom field. §3 is devoted to studying the anyon fields in the space dimension of 2. We show the additivity of the phase exponents in the commutation relations, which restricts possible charges and fluxes of quasiparticles. Some aspects of the anyons in the FQHE are sketched on this basis. In the 1 space dimension, composite field has rather

^{**)} The gas of the H-atoms undergoes the Bose-Einstein condensation. It means that the protons distribute like Bose particles, since the position of them almost coincide with those of the atoms.

clear-cut structure of the commutation relations, which is shown shortly in §4. In §5 we study the hierarchy structure of the FQHS placing emphasis on the special character of the CS gauge field. The composite fermion theory is critically reviewed from this stand point. Finally we make some conceptual remarks on the subject in the last section.

§2. The atom field

The symmetries of the many-body wave function was once investigated by Ehrenfest and Oppenheimer³⁾ under very interesting considerations without referring to the field quantization. We now study the same subject in the framework of the quantum field theory intending to make clear the field aspect of it. We consider here the simplest case of the hydrogen atom.

Let us first introduce the composite field Ψ through the equation

$$\Psi(x_1, x_2) = T\psi(x_1)\phi(x_2) \quad (2)$$

where $\psi(x_1)$ and $\phi(x_2)$ are the elementary fields of the nucleus and the electron respectively in the Heisenberg picture. We, then, define the Bethe-Salpeter amplitude $\langle 0|\Psi(x_1, x_2)|2\rangle$ for the two-particle states and obtain the state of the H-atom by solving the BS equation for it. The nonrelativistic approximation suffices for the present purpose. And we are interested in only the wave function of the ground state, the Fourier transform of which is denoted by $g_{mm'}(\mathbf{k}_1, \mathbf{k}_2)$, where m and m' are the indices of the spin of the nucleus and the electron respectively. We note, however, that these indices are dummy since the spin of the nucleus is frozen in the nonrelativistic situation and therefore the spin freedom of the electron can be neglected in the ground S state.

We next reconstruct the composite field operator by including the bound-state amplitude. If the total and the relative momenta are $K = (K_0, \mathbf{K})$ and $k = (k_0, \mathbf{k})$ respectively, the contribution of the bound state to the annihilation part is given by

$$\begin{aligned} \Psi(X, x) = & C e^{iKX} \sum_{mm'} \int d^3k \\ & \times g_{mm'}(\mathbf{K}, \mathbf{k}) a_m(\eta_1 \mathbf{K} - \mathbf{k}) b_{m'}(\eta_2 \mathbf{K} + \mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{x}), \quad \eta_1 + \eta_2 = 1, \end{aligned} \quad (3)$$

where C is a normalization factor and $a_m(\mathbf{k}_1)$ and $b_{m'}(\mathbf{k}_2)$ are the annihilation operators respectively for the nucleus and the electron which satisfy the anti-commutation relations

$$\{a_m(\mathbf{k}_1), a_{m'}^\dagger(\mathbf{k}_2)\} = \delta_{mm'} \delta(\mathbf{k}_1 - \mathbf{k}_2), \quad \text{etc..} \quad (4)$$

Now, when we observe the stable atom from some great distance we can neglect the scale of

the relative coordinate \mathbf{x} and the atom is represented by the wave function at the origin ^{*)}. We call, after Haag, the neglect of the relative coordinate the space-like asymptotic limit⁴⁾.

The bound state is represented by a local field operator in the space-like asymptotic limit, the annihilation part of which is given by

$$\Psi(X, 0) = CA(\mathbf{K})e^{i\mathbf{K}X}, \quad A(\mathbf{K}) = \int g(\mathbf{K}, \mathbf{k})a(\eta_1\mathbf{K} - \mathbf{k})b(\eta_2\mathbf{K} + \mathbf{k})d^3k, \quad (5)$$

where the dummy spin indices are omitted. $A(\mathbf{K})$ satisfies commutation relations

$$[A(\mathbf{K}), A(\mathbf{K}')] = [A^\dagger(\mathbf{K}), A^\dagger(\mathbf{K}')] = 0 \quad (6)$$

and

$$[A(\mathbf{K}), A^\dagger(\mathbf{K}')] = \delta(\mathbf{K} - \mathbf{K}') \quad (7)$$

$$\begin{aligned} & - \int d^3k g(\mathbf{K}, \mathbf{k})g^*(\mathbf{K}', \eta_2\mathbf{K} - \eta_2\mathbf{K}' + \mathbf{k})a^\dagger(\mathbf{K}' - \eta_2\mathbf{K} - \mathbf{k})a(\eta_1\mathbf{K} - \mathbf{k}) \\ & - \int d^3k g(\mathbf{K}, \mathbf{k})g^*(\mathbf{K}', \eta_1\mathbf{K}' - \eta_1\mathbf{K} + \mathbf{k})b^\dagger(\mathbf{K}' - \eta_1\mathbf{K} + \mathbf{k})b(\eta_2\mathbf{K} + \mathbf{k}) \end{aligned} \quad (8)$$

where the normalization of the momentum-space wave function is assumed to be 1. The spectral condition forbids the last two terms in the right hand side to have matrix elements within the subspace of the bound state.^{*)} We therefore neglect them and reach the canonical commutation relations for the asymptotic atom field.

We have derived the canonical commutation relations for the bound-state fields. This does not, of course, mean the point-like particle picture since $A^\dagger(\mathbf{K})$ create a particle in the plane-wave state. The point-particle picture is relevant only in the interactions.

§3. Anyon fields in the 2 space dimension

The anyon is a particle-like excitation in the 2+1 dimensional system, which is observed, for example, in the phenomena of the FQHE. It is characterized by the fractional statistics in which interchange of two anyons results in any phase of the wave function⁶⁾. The electrons system in the FQHE is confined in some 2 dimensional surface by the complicated electromagnetic interactions with the surrounding materials. The most important effect of the confinement is a change of the topological structure of the configuration space. In the

^{*)} We should use a relativistic equation in deeper classes of the compositeness hierarchy. Then, the wave function at the origin becomes a divergent quantity in some cases. We have to renormalize it⁵⁾.

^{*)} The reason for this is that the operator $b^\dagger b$, for example, in (8) changes the momentum of the electron for $\mathbf{K} \neq \mathbf{K}'$ without changing the momentum of the nucleus and thus the resulting total momentum cannot satisfy the energy-momentum relation of the bound state. For the exceptional case $\mathbf{K} = \mathbf{K}'$ (8) gives a finite correction to $\delta(0)$, which should be neglected.

2+1 dimensional field theory, this boundary condition is settled by allowing the CS gauge term in the Lagrangian, which is a mathematical device to replace the confining interaction and leads to the exotic statistics⁷⁾.

3.1. Exotic statistics in the Chern-Simons field theory

We consider two species of the charged particles the fields of which are denoted by ψ and ϕ . We assume, for definiteness, the bosonic commutation relations among them and call them the background bosons. These bosons interact with the CS fields, for which we introduce three CS terms in the Lagrangian according to Ezawa-Hotta-Iwazaki⁸⁾⁹⁾. Then, the CS part of the Lagrangian becomes

$$\begin{aligned}\mathcal{L}_{\text{CS}} = & (\partial_\mu + ia_\mu)\psi^*(\partial^\mu - ia^\mu)\psi - m^2\psi^*\psi - \frac{1}{4\alpha}\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda \\ & (\partial_\mu + ib_\mu)\phi^*(\partial^\mu - ib^\mu)\phi - M^2\phi^*\phi - \frac{1}{4\beta}\varepsilon^{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda \\ & - \frac{1}{4\gamma}\varepsilon^{\mu\nu\lambda}(a_\mu\partial_\nu b_\lambda + b_\mu\partial_\nu a_\lambda),\end{aligned}\tag{9}$$

where the last term governs the mutual statistics between the fields ψ and ϕ . We quantize this system by following the procedure developed by Semenoff⁷⁾¹⁰⁾. The constraint conditions become

$$\frac{1}{2\alpha}\varepsilon_{ij}\partial_i a_j + \frac{1}{2\gamma}\varepsilon_{ij}\partial_i b_j = j_0,\tag{10}$$

$$\frac{1}{2\gamma}\varepsilon_{ij}\partial_i a_j + \frac{1}{2\beta}\varepsilon_{ij}\partial_i b_j = k_0,\tag{11}$$

where j_0 and k_0 are the 0th component of the current of ψ and ϕ respectively. Under these conditions we get the Hamiltonian in which ψ and ϕ couple minimally to a_i and b_i ($i=1,2$). Further, by assuming the gauge conditions $\partial_i a_i = \partial_i b_i = 0$ we find that the CS fields are given by

$$a_i(x) = \frac{1}{\pi}\partial_i \int d^2y \Omega(\mathbf{x} - \mathbf{y}) \{\mu_a j_0(y) - \mu k_0(y)\} \equiv \partial_i \Theta_a(x),\tag{12}$$

$$b_i(x) = \frac{1}{\pi}\partial_i \int d^2y \Omega(\mathbf{x} - \mathbf{y}) \{\mu_b k_0(y) - \mu j_0(y)\} \equiv \partial_i \Theta_b(x),\tag{13}$$

$$\Omega(\mathbf{x} - \mathbf{y}) = \arctan \frac{x^2 - y^2}{x^1 - y^1}.\tag{14}$$

The coefficients μ 's are given by

$$\mu_a = \frac{\alpha\gamma^2}{\gamma^2 - \alpha\beta}, \quad \mu_b = \frac{\beta\gamma^2}{\gamma^2 - \alpha\beta}, \quad \mu = \frac{\alpha\beta\gamma}{\gamma^2 - \alpha\beta}. \quad (15)$$

We see in (12) and (13) the characteristics of the CS field embodying the boundary condition: We can eliminate the CS gauge interactions from the Hamiltonian by applying the (singular)gauge transformations

$$\psi(x) = \exp(i\Theta_a(x))\hat{\psi}(x), \quad \phi(x) = \exp(i\Theta_b(x))\hat{\phi}(x) \quad \text{etc.} \quad (16)$$

and obtain anyons as follows. By applying (16) the equal-time commutation relations become exotic

$$\begin{aligned} \hat{\psi}(x)\hat{\psi}(y) - e^{i(2n+1)\mu_a}\hat{\psi}(y)\hat{\psi}(x) &= 0, \\ \hat{\psi}(x)\hat{\pi}(y) - e^{i(2n+1)\mu_a}\hat{\pi}(y)\hat{\psi}(x) &= i\delta(\mathbf{x} - \mathbf{y}), \\ \hat{\phi}(x)\hat{\phi}(y) - e^{i(2n'+1)\mu_b}\hat{\phi}(y)\hat{\phi}(x) &= 0, \\ \hat{\phi}(x)\hat{\chi}(y) - e^{i(2n'+1)\mu_b}\hat{\chi}(y)\hat{\phi}(x) &= i\delta(\mathbf{x} - \mathbf{y}), \\ \hat{\psi}(x)\hat{\phi}(y) - e^{i(2n''+1)\mu}\hat{\phi}(y)\hat{\psi}(x) &= 0, \\ &\text{etc.,} \end{aligned} \quad (17)$$

where π and χ are the fields canonical conjugate to ψ and ϕ respectively. The odd integers $2n + 1$, etc. come from the multi-valuedness of the function Ω . Thus, we have the anyon fields $\hat{\psi}(x)$, $\hat{\phi}(x)$, etc.. We note that the exclusion principle holds for anyons.

3.2. Composite anyon field

If we express the free anyon fields as superpositions of the plane waves

$$\hat{\psi}(x) = \int \frac{d^2k}{2\pi} \frac{1}{2\omega} \{a(\mathbf{k})e^{-ikx} + c^\dagger(\mathbf{k})e^{ikx}\}, \quad (18)$$

$$\hat{\phi}(x) = \int \frac{d^2k}{2\pi} \frac{1}{2\omega} \{b(\mathbf{k})e^{-ikx} + d^\dagger(\mathbf{k})e^{ikx}\}, \quad (19)$$

the commutation relations among the operators a , b , c ... becomes

$$\begin{aligned} a(\mathbf{k})a(\mathbf{k}') - e^{i(2n+1)\mu_a}a(\mathbf{k}')a(\mathbf{k}) &= 0 \\ a(\mathbf{k})a^\dagger(\mathbf{k}') - e^{i(2n+1)\mu_a}a^\dagger(\mathbf{k}')a(\mathbf{k}) &= \delta(\mathbf{k} - \mathbf{k}') \\ b(\mathbf{k})b(\mathbf{k}') - e^{i(2n'+1)\mu_b}b(\mathbf{k}')b(\mathbf{k}) &= 0 \end{aligned}$$

$$\begin{aligned}
b(\mathbf{k})b^\dagger(\mathbf{k}') - e^{i(2n'+1)\mu_b}b^\dagger(\mathbf{k}')b(\mathbf{k}) &= \delta(\mathbf{k} - \mathbf{k}') \\
a(\mathbf{k})b(\mathbf{k}') - e^{i(2n''+1)\mu}b(\mathbf{k}')a(\mathbf{k}) &= 0 \\
&\text{etc..}
\end{aligned} \tag{20}$$

When $\mu's/\pi$ are integers we can construct the Fock space and have the particle picture of anyons. We note that the condition by which anyons have the particle interpretation can be relaxed some what as it is discussed in §3.3. and §3.4..

Now, let us assume that two anyon fields with the statistical parameters α and β form a composite system. Relying on 'the glue is unimportant' principle, we can calculate the commutators of the composite operators in the same way as in §2, where the anticommutators are replaced by the exotic ones (20): The composite operator A in the space-like asymptotic limit is defined by

$$A(K) = \int g(K, k) a(\eta_1 \mathbf{K} - \mathbf{k}) b(\eta_2 \mathbf{K} - \mathbf{k}) d^2 k. \tag{21}$$

A satisfies, under the same consideration discussed in §2, the commutation relations

$$A(\mathbf{K})A(\mathbf{K}') - e^{i(2n+1)\mu_{ab}}A(\mathbf{K}')A(\mathbf{K}) = 0, \tag{22}$$

$$A(\mathbf{K})A^\dagger(\mathbf{K}') - e^{i(2n+1)\mu_{ab}}A^\dagger(\mathbf{K}')A(\mathbf{K}) = \delta(\mathbf{K} - \mathbf{K}'), \tag{23}$$

where¹¹⁾

$$\mu_{ab} = \mu_a + \mu_b, \quad \text{mod } 2\pi. \tag{24}$$

3.3. Quasiparticle picture of the anyon

We have assumed that the background fields ψ and ϕ have one unit of the CS(statistical) charge in the Lagrangian (9). The CS flux is, therefore, quantized in the unit 2π ^{*)}. Suppose an object(anyon) a with the CS charge m_a carrying the f_a units of the flux which is perpendicular to the confined surface. If two of such object interchange their position by the half circular motion around each other, the two-particle wave function acquires the phase $\exp(\pm i f_a m_a \pi)$ by virtue of the Aharonov-Bohm effect for the CS flux. By comparing with (17), we are tempted to identify $f_a m_a$ with μ_a/π . However, it is shown to be not always possible by inspecting the composite anyon state: It is natural to assume the conservations of charge and flux in the process of the binding. Then, the charge and the flux of which μ_{ab}

^{*)} We are using the natural unit $\hbar = 1$.

consists are the sums of those possessed by the anyons a and b. But this assignment is not always compatible with (24) as is shown in the following.

We first note that anyonic particles cannot even coexist peacefully if they have different charge/flux ratios, since we cannot define the interchange itself of two anyons consistently for them. Consider, then, n anyons with the flux f_i and the charge m_i which satisfy the relations

$$m_i = k f_i, \quad i = 1, 2, \dots, n \quad (25)$$

with a common constant k . Suppose next that these n anyons form a composite state, which have the flux $\sum f_i$ and the charge $\sum m_i$. The equation (24) then gives the consistency condition

$$k\left\{\left(\sum_{i=1}^n f_i\right)^2 - \sum_{i=1}^n f_i^2\right\} = 2p, \quad p = \text{integer}. \quad (26)$$

For the identical n anyons this condition becomes $n(n-1)f_1m_1 = 2p$.

The anyon does not have the particle picture in general. But, it can be regarded as a particle having the flux f_L and the charge m_L if the converted statistical parameter μ_L satisfies

$$\frac{\mu_L}{\pi} = p(\text{integer}) = f_L m_L, \quad (27)$$

because we can construct the Fock space for this value of the phase. Now, suppose that this anyonic particle decays into n anyons satisfying the criterion (26). We can also regard these products of dissociation to be the quasiparticles, since their parent state has the particle picture.

3.4. Anyons in the fractional quantum Hall effects

The phenomenology of the FQHE becomes simple by using the quasiparticle concepts of anyons when the bosonic background fields are assumed. The ground state (Laughlin state) of the FQHE is an incompressible quantum fluid made of the background boson with the charge $-e$ carrying f_L units of the magnetic flux¹²⁾, where the quantization unit is $\phi_0 = 2\pi/e$. We identify it as an anyonic particle with the statistical parameter α_L given by

$$\frac{\alpha_L}{\pi} = f_L = \frac{B}{\phi_0 \langle \rho \rangle_L}, \quad (28)$$

where $\langle \rho \rangle_L$ is the boson density and B is the external magnetic field. We see that the filling factor ν^L is $1/f_L$ in this state. The CS field associated with the boson is given through (10)

with $\alpha = \alpha_L$, $\gamma = \infty$ and $j_0 = \rho$, which becomes

$$\langle \rho \rangle_L = \frac{1}{2\alpha_L} \varepsilon_{ij} \partial_i \langle a_j \rangle_L. \quad (29)$$

We therefore have $\langle a_i \rangle_L = eA_i$ for the ground Laughlin state, where $\varepsilon_{ij} \partial_i A_j = B$. Thus, the assumed CS field is determined completely by the real external field in this state.

Suppose that an elementary topological vortex is excited in the increasing magnetic field B^8). It is given by the configuration

$$\rho = \langle \rho \rangle_L + \rho_q, \quad a_i = eA_i + v_i, \quad v_i \rightarrow \partial_i \theta (\text{at the infinity}), \quad (30)$$

where θ is the azimuthal angle in the frame whose origin is at the center of the vortex. The CS flux of this excitation becomes 2π , which amounts to a unit flux ϕ_0 of the real magnetic field. On the other hand, the statistical charge Q of the field q is given by

$$Q = \int \rho_q d^2x = \frac{1}{2\alpha_L} \int \varepsilon_{ij} \partial_i v_j d^2x = \frac{\pi}{\alpha_L} = \frac{1}{f_L}. \quad (31)$$

Thus, the vortex is identified with the quasihole(anti-quasiparticle) having the unit flux and the CS charge $m_q = 1/f_L$. We, further, find that the statistical parameter of the field q is given by $\alpha_q = m_q \pi$ because the anyon q is a dissociation product of another anyon having particle nature.

The accumulated quasiholes form the second incompressible-fluid state, where the next vortices are created, which again form the higher Laughlin state, and so on. Though the anyons in this hierarchical structure cannot be regarded as the products of the simple dissociation, they are the quasiparticles with the definite flux and the CS charge as is shown in §5.

§4. A scalar field in the 1 space dimension

In this section, we briefly study the statistics in the 1 space dimension for completeness sake.

The massless scalar field(Nambu-Goldstone boson) in the Schwinger¹³⁾ and the Thirring¹⁴⁾ models was investigated by K.R. Ito¹⁵⁾ and by Nakanishi¹⁶⁾. Ito gave an explicit model of the operators for this composite field. The annihilation operator is given by

$$d^+(p^1) = \int \frac{dq^1}{p^0} \{ \theta(p^1) : \Psi_1^*(q^1) \Psi_1(p^1 + q^1) : + \theta(-p^1) : \Psi_2^*(q^1) \Psi_2(p^1 + q^1) : \}, \quad (32)$$

where

$$\Psi(p^1) = u(p^1)a(p^1) + v(p^1)b^*(-p^1),$$

$$u(p^1) = \begin{pmatrix} \theta(p^1) \\ \theta(-p^1) \end{pmatrix}, \quad v(p^1) = \begin{pmatrix} \theta(-p^1) \\ \theta(p^1) \end{pmatrix}, \quad (33)$$

$\theta(x)$ is the step function and $a(b)$ is the annihilation operator for the original fermion(its antiparticle).*) The defined operator satisfies exactly the canonical commutation relations

$$[d^+(p^1), d^+(p'^1)^*] = \delta(p^1 - p'^1), \quad \text{etc..}$$

This is due to the special character of the 1-dimensional 'spinor' (33).

§5. Hierarchy of the FQHE

We have found in §3.4. that the CS field is equal to the real field in the Laughlin state. This is an advantage of assuming the background field to be bosonic and this mechanism continues to work in higher part of the hierarchy. We will then have two standpoints depending on the ways how to treat the CS field. The first one is developed in the next subsection, where we perform the anyonic gauge transformation to eliminate it at the every stage of the hierarchy. In the second strategy, briefly described in §5.2., the CS fields and the background bosons are all eliminated by the chain relations of replacement.

5.1. The hierarchy in terms of the anyonic gauge transformation

Let us begin with the s th quasiparticle of the density ρ_s which has the charge $q_s e$ and the statistical parameter α_s . The quasiparticles form the Laughlin state with the density $\langle \rho \rangle_{s+1}$. Then, the $s+1$ th vortex is excited, which is given by the configuration

$$\rho_s = \langle \rho \rangle_{s+1} + \rho_{s+1}, \quad a_{i(s)} = \langle a_i \rangle_{s+1} + v_{i(s+1)}, \quad (34)$$

$$v_{i(s+1)} \rightarrow \tau_{s+1} \partial_i \theta(\text{at the infinity}), \quad (35)$$

where the vortex with $\tau_{s+1} = -1(1)$ is identified with the quasihole(quasiparticle). We assume that the frozen part $\langle \rho \rangle_{s+1}$ satisfies

$$\langle \rho \rangle_{s+1} = \frac{1}{2\alpha_s} \varepsilon_{ij} \partial_i \langle a_j \rangle_{s+1}. \quad (36)$$

*) There is an error in the Eq.(5.3) of the original paper. We have corrected the definition of $d^+(p^1)$. See the reference¹⁵⁾ for further details.

Then, the constraint equation gives

$$\rho_{s+1} = \frac{1}{2\alpha_s} \varepsilon_{ij} \partial_i v_{j(s+1)}. \quad (37)$$

Now, the $s + 1$ th CS charge is given by

$$m_{s+1} = \int \rho_{s+1} d^2x = -\tau_{s+1} \frac{\pi}{\alpha_s}. \quad (38)$$

The parameter m_s determines the statistics, which is also governed by the parameter α_s which is defined by μ_a in (15) with $\gamma = \infty$. We, therefore, identify α_s/π with m_s and get the reciprocal relation⁸⁾

$$m_{s+1} = -\frac{\tau_{s+1}}{m_s} + 2p_{s+1}, \quad (39)$$

where p_{s+1} is an integer representing the multivalued nature of the statistical factor.

There act the Coulomb forces among the quasiparticles of (34) and the lowest energy state becomes the incompressible fluid. In order to estimate the charge q_{s+1} of the anyons, we next investigate the structure of this fluid state¹⁷⁾: The N quasiparticles build up the $s + 1$ th Laughlin fluid, the wave function of which is given by

$$\Psi_{s+1} \propto \prod_{i < j} (z_i - z_j)^{m_{s+1}} \exp\left(-\sum_{i=1}^N \frac{|z_i|^2}{4\ell^2}\right), \quad (40)$$

where $z_i = x_i + iy_i$ and the index m_{s+1} comes from the anyonic statistics. $1/2\pi\ell^2$ is the state density of the quasiparticle, which is $|q_s|$ times that of the electron. In the wave function (40) $M = Nm_{s+1}$ is the highest angular momentum of the constituent quasiparticle and the area occupied by the system is given by $2\pi\ell^2 Nm_{s+1}$. Now, the buried quasiparticle shows up itself as the excitation in this fluid. The quasihole excitation is expressed by the operation $S(z_0)$ on the wave function, since it has an unit of the CS flux;

$$S(z_0)\Psi_{s+1} \propto \prod_i (z_i - z_0)\Psi_{s+1}. \quad (41)$$

The area of the system increase by $2\pi\ell^2$ by this operation, which corresponds to the $1/m_{s+1}$ of the constituent s th quasiparticle. Since the quasihole excitation has the charge opposite to the constituent, we have the charge relation

$$q_{s+1} = \tau_{s+1} \frac{q_s}{m_{s+1}}. \quad (42)$$

Getting back to (40), we next note that the number of the one-quasihole states is given by Nm_{s+1} for the large N . Since the constituent quasiparticle has the charge $q_s e$, the number of

the electron-equivalent states(the number of the flux quanta) becomes $Nm_{s+1}/|q_s|$. On the other hand, since the total charge of the quasi-particles is $-\tau_{s+1}Nq_se$, the charge fraction per the electron-equivalent state is $-\tau_{s+1}q_s|q_s|/m_{s+1}$. By considering the change in the charge fraction we have the recurrence formula for the filling factor ν^s

$$\nu^{s+1} = \nu^s + \tau_{s+1} \frac{q_s|q_s|}{m_{s+1}}. \quad (43)$$

We finally obtain the hierarchy of the FQH states from (39), (42) and (43).¹⁸⁾

5.2. *The hierarchy derived from the replacement mechanism*

We have assumed the anyon transformation (16) and eliminated the CS fields at every levels of the hierarchy in §5.1.. Another way to get the hierarchy of the filling factors is working throughout with the background bosons. We have chain relations including the CS fields in this case and all unphysical fields(the bosons and the CS fields) disappear through successive replacement. Such an approach was pursued by Ezawa, Hotta and Iwazaki⁸⁾ in the framework of a nonrelativistic model. They obtained the hierarchy in which the filling factors are represented by continued fractions¹⁹⁾.

5.3. *The limit of even denominators; neutral Fermi-gas states*

It was argued that the composite fermion theory²⁰⁾ predicts the Fermi liquid states at the even-denominator filling factors $\nu = 1/2m^{21)}$. And the existence of these states has been regarded as the most powerful evidence for this theory. However, it should be noted that the noninteracting fermions states are also deduced from the limit of some hierarchy series in the background-boson gauge.

Let us look for the simplest case of $\nu = 1/2$: If we put $\tau_s = 1$, $p_1 = 2$ and $p_s = 1(s > 1)$ in the recurrence relations (39), (42) and (43) we get

$$m_s = \frac{2s+1}{2s-1}, \quad q_s = \frac{1}{2s+1} \quad \text{and} \quad \nu^s = \frac{s}{2s+1}. \quad (44)$$

Thus, the $s \rightarrow \infty$ limit of the series gives a state of gas consisting of the neutral fermions.

The origin of the Fermi gas state is essentially different from that of the Fermi liquid which is predicted by the composite fermion theory. In the latter, it is said that the external magnetic field is cancelled by the CS field and then the fermions become noninteracting except for the mutual interactions among themselves. In contrast with it, the above anyons become noninteracting because they are neutral. In the composite fermion theories, it must have been overlooked that the CS field is a fictitious field.

§6. Discussion

We have shown the canonical commutation relations for the 3-dimensional bound-state field. It is not persuadable to interpret the statistic property of the bound state by using the particle quantum mechanics. Instead, we should first conceive of the composite field consisting from the constituent fields as a nature being acquired by the space-time points. Translation of it into the language of the particle quantum mechanics may be that ‘the constituents loose their individuality in the bound state and behave as a quantum mechanical unity’²²⁾.

In the hierarchy of the compositeness, the Cooper pair of the super-conductor may be in the highest class. The annihilation operator for it is written as

$$\Phi = \sum_{\mathbf{k}} A(\mathbf{k}) b_{\uparrow}(\mathbf{k}) b_{\downarrow}(-\mathbf{k}).$$

We then have

$$[\Phi, \Phi] = 0 \quad \text{and} \quad [\Phi, \Phi^{\dagger}] = 1$$

under the similar restriction leading to (7) where the terms in (8) which represent possible decays of the bound state are neglected. We have seen that a composite field in the 1 space dimension satisfies the canonical commutation relations without restricting the Fock space of the original fermion. This is consistent with the stability of the Nambu-Goldstone boson.

The difference between the field theory and the many-particle quantum mechanics is more prominent in the 2 space dimension. The quantized anyon fields do not generally have the Fock-space representation. Therefore, we cannot conceive of the particle picture of the anyon except for the very special cases. We may at best define the anyon or quasiparticle as an object having the definite flux and charge. However, we do not have the Fock space for this quasiparticle field, that is, the number of quasiparticles is not generally a good quantum number. In such special cases as the FQHS the number of anyons becomes definite.

For the composite state of two anyons, we have shown the proper anyonic commutation relations. The phase exponents in the commutation relations of the constituent anyons add up to that of the composite anyon, as it is expected. This relation gives some restrictions for the CS charge and flux of the anyon which is the decay product of a quasiparticle anyon.

The complex nature of the anyon comes from the CS gauge interaction, which mathematically represents the plane(surface) geometry of the system. The advantage of the background-boson gauge is that the statistical parameters and the CS fields at each stage of the FQHS are determined by the physical parameters and field. Further, the background bosons which have been introduced as fictitious fields convert themselves into the physical

anyons by virtue of the singular gauge transformation. Then, the commutation relations impose the exclusion principle, except for a special case of the bosonic anyon, on the many-body states of the anyons. The generalized Laughlin states are formed as a consequence of this principle(an analogue of the Fermi degeneracy).

On the other hand, if we take the background-fermion gauge the CS fields are not thoroughly determined by the physical field in the FQHE and therefore the fictitious fields remain in the very final results. This reduces our confidence in interpreting them. Though the CS field just cancels the external field for the half filling state, it remains still fictitious. The real reason for the disappearance of the interaction can not be this cancellation.

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